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In quest of weak essentially undecidable theories.

(Some conjectures)

Kurt Gödel was the first who proved (in Gödel[1931]) the undecidability of the arithmetic of natural numbers. He dealt with a very strong formal system which contained the arithmetic. His result has been improved by several authors. Finally Mostowski, Robinson and Tarski presented together (in the booklet: Tarski[1953]) a very weak arithmetic Q with finite set of elementary axioms which is essentially undecidable. Q was studied previously by R.M. Robinson (in Robinson[1950]).

The above mentioned researches belong to the stream or currant in meta-mathematical studies, which was inspired by the intuition related to the conception of numbers. There was also in XX century another stream in metamathematics. The researchers such as S. Leśniewski, W.O. Quine and partly also A. Tarski, A.M. Turing and A.A. Markov have been rather inspired by the intuition related to the conceptions of symbols, formulas, and in general by the intuition of meaningful texts. The most general theory of texts was proposed by Tarski[1933] as a theory of concatenation.

Last years I tried to get some results, which have been obtained in the first stream, but using purely the methods which seem to be appropriate to the second stream. In the paper Grzegorczyk[2005] I have proved that a very weak theory of texts called TC, which is in fact the theory of concatenation formulated by Tarski in Tarski[1933] is undecidable. The proof I have applied is perhaps more intuitive and more elementary, than the proofs which have been used by the previous authors, but cannot be enhanced to a proof of essential undecidability. However I dare conjecture that the theory TC is essentially undecidable.

The conjectures concerning essential undecidability may be voiced also for other very weak theories which till now have not been studied.

One of the methods of finding such theories consists of applying relations instead of functions (as primitive notions) and writing explicitly the axioms of uniqueness and of existence (of the result of the operations considered as primitives). These axioms usually are hidden, covered by the graphical style of writing functions.

In the following I would like to raise the problem of essential undecidability for two very weak theories. The first will be an arithmetic weaker than the theory Q of R.M. Robinson, and the second: a theory of concatenation weaker than the theory TC. The easy proofs that the considered theories are really weaker (the axioms which seem to be not necessary for undecidability are independent) are very modest results (and the sole results) of the paper.
1. Arithmetic weaker than Q

Let’s consider the following theory $Q^\ast$ which has 4 primitives: 0 (a constant), $S$ (the unique conception which is a function), and: $\Delta$, $\times$ (which are ternary relations). The theory $Q^\ast$ is an elementary arithmetic of natural numbers which has the following 11 axioms:

Aq1  $0 \neq Sx$
Aq2  $x=0 \lor \exists y \; x=Sy$
Aq3  $Sx=Sy \rightarrow x=y$
Aq4  $\Delta(x,0,x)$
Aq5  $\Delta(x,y,z) \rightarrow \Delta(x,Sy,Sz)$
Aq6  $\forall x,y \; \exists z \; \Delta(x,y,z)$
Aq7  $(\Delta(x,y,z) \land \Delta(x,y,v)) \rightarrow z=v$
Aq8  $\times(x,0,0)$
Aq9  $(\times(x,y,z) \land \Delta(x,z,u)) \rightarrow \times(x,Sy,u)$
Aq10  $\forall x,y \; \exists u \; \times(x,y,u)$
Aq11  $(\times(x,y,z) \land \times(x,y,v)) \rightarrow z=v$

The existence and uniqueness of the arithmetical sum are explicitly written in the axioms: Aq6 and Aq7 and the existence and uniqueness of the arithmetical product are also explicitly expressed in the axioms Aq10 and Aq11. Only the existence and uniqueness of the successor are (‘perfidiously’) hidden by writing “$S$” as a symbol for a function.

According to the general logical rules (of the first order functional calculus) – we are able, of course, to introduce in the above presented theory $Q^\ast$ the two following definitions of functions $+$ and $\cdot$:

\[
\begin{align*}
x+y=z & \equiv \Delta(x,y,z) \\
x\cdot y=z & \equiv \times(x,y,z)
\end{align*}
\]

then from the axioms: Aq4 – Aq11 we get the recursive equations appropriate to addition and multiplication:

\[
\begin{align*}
x+0 & = x \\
x+Sy & = S(x+y) \\
x\cdot 0 & = 0 \\
x\cdot Sy & = x+x\cdot y
\end{align*}
\]

Then we realize that $Q^\ast$ is really very similar to the theory $Q$ studied for the first time by Raphael M. Robinson in 1950 and exhibited in the booklet Tarski [1953]. The theories $Q$ and $Q^\ast$ are different. (Multiplication in these theories is not commutator.)

One may hope that the existential axiom hidden in the functional character of “$S$” suffices for the proof of undecidability and then the axioms Aq6 and Aq10 may be canceled. Hence I would like to show that these axioms are independent. And indeed:

Theorem 1. A. The existential axiom Aq10 is independent on the rest of the axioms: Aq1 – Aq11
B. The existential axioms Aq6 and Aq10 are together independent on the rest of the axioms (this means on of the axioms (Aq1 – Aq5, Aq7 – Aq9, Aq11) of the theory $Q^\ast$.

Proof: Look at the following model: $\langle M,0,S,\Delta,\times \rangle$.
I assume the following notation:

$I=$ integers.
$J=\{1,x\}$ for all $x \in I$.  $N=\{0,n\}$ for all $n \in I$ such that $n \geq 0$. 

2
\(M = N \cup J.\)

(N is the standard part of the model. J is the non standard part.)

We can imagine \(M\) as linearly-ordered:
\[
\langle 0,0 \rangle, \langle 0,1 \rangle, \langle 0,2 \rangle, \ldots, \langle 1,-2 \rangle, \langle 1,-1 \rangle, \langle 1,0 \rangle, \langle 1,1 \rangle, \langle 1,2 \rangle, \ldots
\]
in the order-type \(\omega^+\omega^*\omega^+\).

We put:
\[
\begin{align*}
(1) & \quad 0 = \langle 0,0 \rangle \\
(2) & \quad S\langle k,x \rangle = \langle k,x+1 \rangle \text{ for each pair } \langle k,x \rangle \in M.
\end{align*}
\]

Hence the axioms Aq1 - Aq3 (for successor) are obviously satisfied.

First we concentrate on the part A. of the Theorem.

Let's consider the following interpretation of the relations \(\Delta\) of addition:
\[
\text{D1 } \Delta(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle) \equiv (n=k \land z=x+y),
\]
for any \(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle \in M.\)

According to D1 we easy realize that
\[
(3) \quad \Delta(\langle k,x \rangle,\langle 0,0 \rangle,\langle k,x \rangle).
\]
This means that Aq4 is satisfied. Also:
If \(\Delta(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle)\) then \(z=x+y\) and \(z+1=x+(y+1)\) and this implies by D1 that \(\Delta(\langle k,x \rangle,\langle m,y+1 \rangle,\langle n,z+1 \rangle)\). Hence Aq5 is satisfied. Also putting for \(z\) \(x+y\) we get from D1:
\[
\Delta(\langle k,x \rangle,\langle m,y \rangle,\langle x+y \rangle)
\]
This shows that Aq6 is satisfied.

If \(\Delta(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle)\) and \(\Delta(\langle k,x \rangle,\langle m,y \rangle,\langle n,z' \rangle)\) then by D1 \(n=k\) and \(n'=k\) and \(z=x+y\) and also \(z'=x+y\). Thus \(n=n'\) and \(z=z'\), and \(\langle n,z \rangle = \langle n,z' \rangle\).
This shows that Aq7 is satisfied. All axioms concerning addition are satisfied.

Let's add the following interpretation of the relation \(\times\):
\[
\text{D2 } \times(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle) \equiv ((m=0 \land y \geq 0 \land z=xy) \land (y=0 \land n=0) \lor (y>0 \land n=k)),
\]
for any \(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle \in M.\)

According to D2 if \(y=0\) then \(n=0\) and \(z=0\). Hence for any \(\langle k,x \rangle \in M:\)
\[
\times(\langle k,x \rangle,\langle 0,0 \rangle,\langle 0,0 \rangle).
\]
This shows that the axiom Aq8 is satisfied.

In order to check Aq9 suppose that:
\[
(4) \quad \times(\langle k,x \rangle,\langle m,y \rangle,\langle n,z \rangle) \quad \text{and}
(5) \quad \Delta(\langle k,x \rangle,\langle n,z \rangle,\langle p,u \rangle).
\]
We shall prove that \(\times(\langle k,x \rangle, S\langle m,y \rangle,\langle p,u \rangle).\) From (4) and D2 we get that:
\[
\begin{align*}
(6) & \quad (m=0 \land y \geq 0 \land z=xy) \land \quad \text{and} \\
(7) & \quad (y=0 \land n=0) \lor (y>0 \land n=k),
\end{align*}
\]
from (5) and D1 we know that:
\[
\begin{align*}
(8) & \quad k=p \\
(9) & \quad u=x+z.
\end{align*}
\]
(6) and (9) imply that:
\[
(10) \quad u=x+x=xy.
\]
If \(y \geq 0\) (according to (6)) then also:
(11) $Sy > 0$

From (6), (10) and (11) we get:

(12) $m=0 \land Sy \geq 0 \land u=x(Sy)$.

(11) and (8) give that: $(Sy>0 \land p=k)$ and of course also:

(13) $((Sy=0 \land p=0) \lor (Sy>0 \land p=k))$

According to D2 the conjunction of (12) and (13) means that:

(14) $x(\langle k, x \rangle, \langle m, Sy \rangle, \langle p, u \rangle)$.

But according to (2) the formula (14) means that: $x(\langle k, x \rangle, S \langle m, y \rangle, \langle p, u \rangle)$. Hence we have proved that the axiom Aq9 is satisfied in the model.

From the definition D2 it is very easy to see that the axiom Aq11 is satisfied. Only the axiom Aq10 is not satisfied, because we have decided in D2 that the second argument of the relation $\times$ should be standard. Hence for any non standard element $\langle 1, y \rangle$ it holds that:

$\neg x(\langle 1, y \rangle, \langle n, z \rangle)$.

Thus the axiom Aq10 is independent on all other axioms of the theory $Q^\ast$.

To prove the part B. of our Theorem we consider the following model: $\langle M', 0, S, \Delta, \times \rangle$ which is similar to the previous one. We add to the set $M$ one more collection $J'$ of non standard elements:

$M' = M \cup J' = N \cup J \cup J'$, where $J' = \{\langle 2, x \rangle\}$ for all $x \in I$.

Then we adopt the same definitions for $0$, $S$ and $\times$ changing only $M$ into $M'$.

For addition we change a little more:

D1 $\Delta(\langle k, x \rangle, \langle m, y \rangle, \langle n, z \rangle) \equiv ((n=k \land z=x+y) \land (k=1 \equiv m \not= 2) \land (k=2 \equiv m \not= 1))$

for any $\langle k, x \rangle, \langle m, y \rangle, \langle n, z \rangle \in M'$.

The condition $((n=k \land z=x+y) \land (k=1 \equiv m \not= 2) \land (k=2 \equiv m \not= 1))$ means that we do not allow to add the elements of $J'$ to the elements of $J$. Hence it holds that:

$\neg \Delta(\langle 1, x \rangle, \langle 2, y \rangle, \langle n, z \rangle)$

Thus Aq6 is not satisfied in the model. All other axioms are satisfied for the same reason as in the proof of the part A. of the Theorem.

Thus the following theory $Q^\ast$ ($Q^\ast$ is the set of consequences of the axioms {Aq1 – Aq3, Aq4, Aq5, Aq8, Aq9}) is proved to be weaker than $Q^\ast$.

$Q^\ast$ is intuitively interesting because it has yet an intuitive arithmetical meaning though it has so simple non standard models. And it is natural to conjecture that:

C1 $Q^\ast$ is essentially undecidable?
2. A theory of texts weaker than TC

Recently I have proved that the following elementary theory of texts TC is undecidable:

\[ A_1 \quad x^*(y^z) = (x^y)^z \]
\[ A_2 \quad x^y = z^u \rightarrow ((x = z \land y = u) \lor \exists w((x^w = z \land w^u = y) \lor (z^w = x \land w^y = u))) \]
\[ A_3 \quad \neg(x = x^y) \quad A_4 \quad \neg(\beta = x^y) \quad A_5 \quad \neg(\alpha = \beta). \]
(The asterisk denotes concatenation. \( \alpha, \beta \) are two constant elements.) A1 and A2 are due to A. Tarski. As I have mentioned at the beginning I have conjecture that:

C2 TC is essentially undecidable?

If we assume that: \(! (x, y, z) \equiv x^y = z\) then TC may be formalized as having the following axioms:

\[ B_1 \quad ! (y, z, u) \land ! (x, u, v) \land ! (x, y, w) \land ! (w, z, t) \rightarrow v = t \]
\[ B_2 \quad ! (x, y, t) \land ! (z, u, t) \rightarrow ((x = z \land y = u) \lor \exists w(! (x, w, z) \land ! (w, u, y)) \lor (! (z, w, x) \land ! (w, y, u))) \]
\[ B_3 \quad \neg ! (x, y, \alpha) \quad B_4 \quad \neg ! (x, y, \beta) \quad B_5 \quad \neg (\alpha = \beta) \]
B6 \forall x, y \exists t ! (x, y, t)  
B7 ! (x, y, t) \land ! (x, y, v) \rightarrow t = v.

It seems that one may weaken the existential axiom B6 and assume instead B6 the following pair of axioms: B61 and B62:

\[ B_{61} \quad \forall x \exists t ! (x, \alpha, t) \quad B_{62} \quad \forall x \exists t ! (x, \beta, t) \]
B61 and B62 allow only to concatenate \( \alpha \) or \( \beta \) to a given text \( x \).

Theorem 3 The existential axiom B6 is independent on the axioms: B1 – B5, B61, B62, and B7.

Proof. Concatenation of texts is almost the same as the well known set-theoretical conception of: union of order-types (sometimes called simply the addition of orders). The most general conception of text seems to be the following:

**A text is the type of a “colored” ordered set**

The sense of this metaphor will be clear in the following example.

We construct a model \( \langle M, \!, \alpha, \beta \rangle \) for the above theory.

Let consider the well ordering \( < \) of the type: \( \omega + \omega \). Ordinals may be considered as sets such that:

\[ k < n \equiv k \in n. \]

The set M of the elements of the model is defined as follows:

\[ < k, f > \in M \equiv (k \in \omega + \omega \land f \text{ is a mapping of the set } k \text{ on the pair } \langle 0, 1 \rangle). \]

The elements 0, and 1 are arbitrary chosen.

(The mapping \( f \) determines the “coloring” of the elements of the set \( k \).)
We can fix conventionally e.g. that for any element $x \in k$:

$f(x) = 0$ means $x$ is red and $f(x) = 1$ means $x$ is green.

The interpretation of the constants of TC is the following:

$\alpha = \langle 0, 0 \rangle$ where $0$ is the function which is constant: $0(x) = 0$ for any argument $x$.

$\beta = \langle 0, 1 \rangle$ where $1$ is also a constant function: $1(x) = 1$ for any argument $x$.

The relation of concatenation is defined in the following definition:

$(\langle k, f \rangle, \langle n, g \rangle, \langle p, h \rangle) \equiv (p = k + n \land \forall x \in p \ (h(x) = f(x) \text{ for } x \in k \land h(x) = g(x-k) \text{ for } x \geq k))$ where $\langle k, f \rangle, \langle n, g \rangle, \langle p, h \rangle \in M$.

E.g. $\langle \alpha, \beta, \langle 2, h \rangle \rangle$ where $h(0) = 0$ and $h(1) = 1$.

The first part “$p = k + n$” of the definiens of the above definition means that the type of order $p$ is the union $k + n$ of two orders which are concatenated (ordinals are in the same time types of ordering). The second part of the definiens of the above definition determines that the function $h$ which colors the union $p$ preserves the colorings $f$ and $g$ of the components $k$ and $n$ of the union $p$.

We easily verify that in the model $\langle M, \alpha, \beta, ! \rangle$ it is true that:

$\neg (\langle \omega, f \rangle, \langle \omega, g \rangle, \langle p, h \rangle)$

because $\omega + \omega \notin M$ ($\omega + \omega \notin \omega + \omega$). Hence the axiom B6 is not true in the considered model. But all other axioms B1 – B5, B6_1, B6_2 and B7 are true.

Hence we can also conjecture that:

C3 The theory TC* (B1 – B5, B6_1, B6_2, B7) is essentially undecidable.

(On the other hand one can conjecture that the theory of one-color texts is decidable.)

Bibliography


